DUAL DESCRIPTION OF A SPACETIME IN THE EINSTEIN AND EINSTEIN–CARTAN THEORIES

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A new exact ("bouncing") solution to the Einstein–Cartan theory presented, is recast in the standard form of Einstein's theory with the introduction of an effective energy–momentum tensor. It turns out that the latter describes a non-perfect (viscous) fluid, whose energy density becomes negative near the bounce.

The apparent success of gauge field theories has encouraged renewed interest in the Einstein–Cartan (EC) theory as the gauge theory of the Poincaré group [1]. The new element over Einstein's version of general relativity is the introduction of torsion in the spacetime manifold as a *classical* representation of spin. The two theories are formally equivalent [1,2]. This is due to the fact that the algebraic relation between spin and torsion in the EC theory allows one to recast the EC equations in the form of Einstein's field equations. In this letter we present an explicit example of the above equivalence on the basis of an exact non-singular ("bouncing") solution to the EC equations which we have found. More specifically we start with a Bianchi type V cosmological model in the EC theory, containing a perfect spinning fluid and obtain an exact non-singular solution to the field equations with a bounce of the model occurring at $t = 0$. Subsequently we find the effective energy–momentum tensor in the "equivalent" Einstein picture. It turns out that it describes a viscous fluid whose energy density becomes negative near the bounce, a fact which underlines the physical *inequivalence* of the two theories. Our notation is that of Tsoubelis [3].

The perfect-fluid content of the EC spacetime has energy density ρ_c, total pressure $p_c = (1-\gamma)\rho_c$ and spin density $S_{\alpha\beta}$ with $S_{\alpha\beta}u^\beta = 0$, where u^α is the fluid's four-velocity, taken orthogonal to the hypersurface of homogeneity. We further choose the components of $S_{\alpha\beta}$ to vanish except $S_{23} = -S_{32} =: 2\omega$. The torsion is then given by $T^\gamma{}_{\alpha\beta} = S_{\alpha\beta}u^\gamma$ and the energy–momentum tensor by $t_{\alpha\beta} = \mathrm{diag}(\rho_c, p_c, p_c, p_c)$. All quantities refer to the orthonormal tetrad $\{\omega^\alpha\}$ with $\omega^0 = \mathrm{d}t$, $\omega^1 = a\sigma^1$, $\omega^2 = a(\lambda\sigma^2 + \nu\sigma^3)$, $\omega^3 = \mu a\sigma^3$, with $g_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ and a, λ, μ, ν functions of the cosmic time t. $\{\sigma^i\}$ is an invariant one-form basis pertinent to the Bianchi type V group of motions with $\mathrm{d}\sigma^1 = 0$, $\mathrm{d}\sigma^2 = \sigma^1 \wedge \sigma^2$, $\mathrm{d}\sigma^3 = \sigma^1 \wedge \sigma^3$. The field equations can now be solved exactly. Introducing the new time τ by $\mathrm{d}\tau = a^{-3}\mathrm{d}t$ we obtain:

$$\lambda^2 = \mu^{-2} = A^{-1}\Lambda\Omega(1 + \Sigma\Omega^{-1}\sin 2\sqrt{A}\,\tau)\,, \tag{1}$$

$$\Lambda\nu = (2\Omega\Lambda - A\lambda^2 - \Lambda^2\lambda^{-2})^{1/2} - N\lambda\,, \tag{2}$$

$$\tau = \int (a^4 + Ma^{3\gamma} - \tfrac{1}{3}A)^{-1/2}\,\mathrm{d}\ln a\,, \tag{3}$$

$$A = \Omega^2 - \Sigma^2\,. \tag{4}$$

The meaning of the constants A, Λ, M, N, Σ, Ω will become apparent in a moment. We first have the conservation laws for the energy density ρ_c, the shear σ^2 and the spin density ω^2:

$$\tfrac{1}{3}\rho_c a^{3(2-\gamma)} = M\,, \tag{5}$$

$$\sigma^2 a^6 = \Sigma^2\,, \tag{6}$$

$$\omega^2 a^6 = \Omega^2\,. \tag{7}$$

The condition $A > 0$, necessary for the bounce ($A < 0$ gives a singular solution), is fulfilled as long as spin dominates over shear [cf. eqs. (4), (6), (7)]. For the bounce "radius" at $t = 0$ we can write $a_0 = (\frac{1}{3}AM^{-1})^{1/(3\gamma)}$ in terms of which we obtain:

$$\rho_c = (a/a_0)^{3\gamma}(\omega^2 - \sigma^3) . \quad (8)$$

For the special choice $\Lambda^2 = A, N = 0$ our solution reduces to an already known result [4]. However a more natural choice is possible, namely

$$\Lambda = A\Omega^{-1} , \quad N^2 = A\Sigma^2\Omega^{-2} , \quad (9)$$

which gives the model asymptotic open Friedmann behavior, with $\lambda, \mu \to 1$ and $\nu \to 0$ as $|t| \to \infty$.

The effective energy–momentum tensor $T_{\alpha\beta}$ in the Einstein picture, defined by $R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}T_\gamma{}^\gamma g_{\alpha\beta}$ with $R_{\alpha\beta}$ the riemannian Ricci tensor, has the following non-vanishing components:

$$\begin{aligned} &T_{00} = \rho_c - \omega^2 , \quad T_{11} = p_c - \omega^2 , \\ &T_{22} = p_c - (3 - 2\Lambda\Omega^{-1}\lambda^{-2})\omega^2 , \\ &T_{33} = p_c - (2\Lambda\Omega^{-1}\lambda^{-2} - 1)\omega^2 , \\ &T_{23} = T_{32} = -2N\Omega^{-1}(\Lambda N^{-1}\nu\lambda^{-1} + 1)\omega^2 . \end{aligned} \quad (10)$$

The above result is better understood when compared with the general expression for the energy–momentum tensor of a non-perfect (i.e., viscous) fluid [5]:

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p h_{\alpha\beta} + \pi_{\alpha\beta} + u_\alpha q_\beta + u_\beta q_\alpha , \quad (11)$$

where ρ is the total energy density of the fluid, p the scalar and $\pi_{\alpha\beta}$ the anisotropic pressure and q_α the energy flux density. Comparing (10) and (11) we can make the following identifications:

$$\begin{aligned} &\rho = \rho_c - \omega^2 , \quad p = p_c - \omega^2 , \\ &\pi_{22} = -\pi_{33} = 2(\Lambda\Omega^{-1}\lambda^{-2} - 1)\omega^2 , \\ &\pi_{23} = \pi_{32} = -2N\Omega^{-1}(\Lambda N^{-1}\nu\lambda^{-1} + 1)\omega^2 . \end{aligned} \quad (12)$$

The rest of the $\pi_{\alpha\beta}$ components as well as the energy flux density vanish.

Clearly, the energy density of the fluid described by the effective energy–momentum tensor in the Einstein picture has been reduced by ω^2 from the original EC value ρ_c. This is precisely the reason why it may assume negative values and thus allow the avoidance of a singularity [1]. The bounce "starts" at $a = (1 - \Sigma^3\Omega^{-2})^{-1/(3\gamma)}a_0$ at which point the energy density ρ becomes zero (going into negative values) with a symmetric behaviour around $t = 0$ (where $\rho = -\sigma^2$). Similar behaviour is exhibited by the pressure p. Here the term $-\omega^2$ indicates the presence of bulk viscosity. Finally, the usual phenomenological relation $\pi_{\alpha\beta} = -\kappa\sigma_{\alpha\beta}$ cannot be realized with the viscosity coefficient κ a scalar, because of the relation $\pi_{\alpha\beta}\sigma^{\alpha\beta} = 0$ which holds here.

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